


Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year		Final Term Exam Date: January 12, 2016 Course: Mathematics 1 – A Duration: 3 hours
<ul style="list-style-type: none"> The Exam consists of one page Answer All Questions 	<ul style="list-style-type: none"> No. of questions: 4 Total Mark: 100 	
<u>Question 1</u>		
(a) Find y' from the following:		12
(i) $y = 2x^3 + 3x^2 - 3$	(ii) $y = \cos x^2 \cdot \sec x$	
(iii) $y = \ln \sin x + \sin \ln x$	(iv) $y = \tan^{-3} x + \log x$	
(v) $2^y + 2^x = \log(x + y)$	(vi) $y = t \sin t, x = t + \ln t$	
(b) Find the following limits:		
(i) $\lim_{x \rightarrow \pi} (\cot x + \csc x)$	(ii) $\lim_{x \rightarrow 0} \frac{\ln(1 + x^2)}{3^x - 2^x}$	(iii) $\lim_{x \rightarrow 0} \frac{x - \tan x}{x^3 + x^2}$
	(iv) $\lim_{x \rightarrow \infty} \frac{x^8 - 2^x}{x^8 + 3^x}$	8
<u>Question 2</u>		
(a) Write the Maclaurin's series of the function: $f(x) = x \sin x^3$.		4
(b) State and verify the mean value theorem, $f(x) = x - \frac{1}{x}$ in interval $[1, 2]$.		4
(c) Sketch the curve of each function :		
(i) $f(x) = \frac{x}{\sqrt{x^2 - 1}}$	(ii) $g(x) = \frac{x}{1 + x^2}$	12
(d) Find the integrals:		
$\int (x^3 + 3^x) dx, \int (2^x - 3^x)^2 dx, \int \cos^5 2x dx, \int_0^\pi \sin^{10} x dx$		10
<u>Question 3</u>		
(a) Find the sum of the series : $\sum_{r=1}^n \frac{1}{(2r-1)(2r+1)}$		10
(b) Expand the fraction $\frac{3x+5}{x^3 - x^2 - x + 1}$ into ascending power series of x .		10
(c) Find the real and imaginary part of : $\sin z$ and e^{iz} for any complex number z .		5

Question 4	
(a) By Gauss method, solve the system : $x + y + 2z = 7, \quad 2x + y = 8, \quad x + 2y + 2z = 9.$	10
(b) Find the eigenvalues and eigenvector of : $A = \begin{bmatrix} -2 & -2 & -4 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}$	10
(c) Find the value of k in : $12x^3 - 8x^2 + kx + 18 = 0$ given that sum of two roots equal zero, then solve it.	5

Good Luck

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Answer

Answer of Question 1

(a)(i) $y' = 6x^2 + 3x^2 \cdot \ln 3 \cdot 2x - 0$

(ii) $y' = \cos x^2 \cdot \sec x \cdot \tan x - \sin x^2 \cdot 2x \cdot \sec x$

(iii) $y' = \frac{\cos x}{\sin x} + \cos \ln x \cdot \frac{1}{x}$

(iv) $y' = -3 \tan^{-4} x \cdot \sec^2 x + \frac{1}{\ln 10} \cdot \frac{1}{x}$

(v) $2^y \ln 2 \cdot y' + 2^x \ln 2 = \frac{1}{\ln 10} \cdot \frac{1+y'}{x+y}$

Then $y' \left(2^y \ln 2 - \frac{1}{\ln 10} \cdot \frac{1}{x+y} \right) = \frac{1}{\ln 10} \cdot \frac{1}{x+y} - 2^x \ln 2$

Then $y' = \frac{\frac{1}{\ln 10} \cdot \frac{1}{x+y} - 2^x \ln 2}{2^y \ln 2 - \frac{1}{\ln 10} \cdot \frac{1}{x+y}}$

(vi) $y' = \frac{\dot{y}}{\dot{x}} = \frac{t \cos t + \sin t}{1 + \frac{1}{t}}$

-----12-Marks

$$(b)(i) \lim_{x \rightarrow \pi} (\cot x + \csc x) = -\infty + \infty$$

Using L'Hopital's rule, we get

$$\lim_{x \rightarrow \pi} \left(\frac{\cos x}{\sin x} - \frac{1}{\sin x} \right) = \lim_{x \rightarrow \pi} \left(\frac{\cos x - 1}{\sin x} \right) = \lim_{x \rightarrow \pi} \left(\frac{-\sin x - 0}{\cos x} \right) = \frac{0}{-1} = 0$$

$$(ii) \lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{3^x - 2^x} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{\frac{\ln(1+x^2)}{x^2} \cdot x}{2^x \frac{((3/2)^x - 1)}{x}} = \lim_{x \rightarrow 0} \frac{x \frac{\ln(1+x^2)}{x^2}}{2^x \frac{((3/2)^x - 1)}{x}} = \frac{0}{1} \cdot \frac{1}{\ln(3/2)} = 0$$

$$(iii) \lim_{x \rightarrow 0} \frac{x - \tan x}{x^3 + x^2} = \frac{0}{0}$$

Using L'Hopital's rule, we get

$$\lim_{x \rightarrow 0} \frac{x - \tan x}{x^3 + x^2} = \lim_{x \rightarrow 0} \frac{1 - \sec^2 x}{3x^2 + 2x} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{-2\sec^2 x \cdot \tan x}{6x + 2} = \frac{-2(1)(0)}{0 + 2} = 0$$

$$(iv) \lim_{x \rightarrow \infty} \frac{x^8 - 2^x}{x^8 + 3^x} = \frac{\infty - \infty}{\infty + \infty} = \lim_{x \rightarrow \infty} \frac{\frac{x^8}{3^x} - \left(\frac{2}{3}\right)^x}{\frac{x^8}{3^x} + 1} = \frac{0 - 0}{0 + 1} = 0$$

-----8-Marks

Answer of Question 2

$$(a) \text{Since } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

$$\text{Then } \sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} \dots$$

$$\text{Then } f(x) = x \sin x^3 = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} \dots$$

-----4-Marks

(b) The mean value theorem.

We see that $f(x) = x - \frac{1}{x}$ and its derivative $f'(x) = 1 + \frac{1}{x^2}$ are continuous in the given interval $[1, 2]$.

Then $f'(c) = 1 + \frac{1}{c^2} = \frac{f(2)-f(1)}{2-1} = \frac{\frac{3}{2}-0}{1}$. Then $c^2 = 2$ and $c = \pm\sqrt{2}$.

We see that $c = \sqrt{2}$ in $[1, 2]$. Then, the theorem is verified.-----4-Marks

(c)(i) $f(x) = \frac{x}{\sqrt{x^2-1}}$. Domain f is $\mathbb{R} - (-1, 1)$.

▪ Asymptotic lines

The two lines $x = 1$ and $x = -1$ are vertical asymptotes.

Since $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2-1}} = 1$, $\lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2-1}} = -1$

Then, the two lines $y = 1$ and $y = -1$ are horizontal asymptotes.

No inclined asymptotes.

▪ Points of intersection

With x - axis: solve $\frac{x}{\sqrt{x^2-1}} = 0$, we get $x = 0$ outside the domain, No points

With y - axis: $f(0) = \frac{0}{\sqrt{0-1}} = 0$, but $x = 0$ outside the domain, No points.

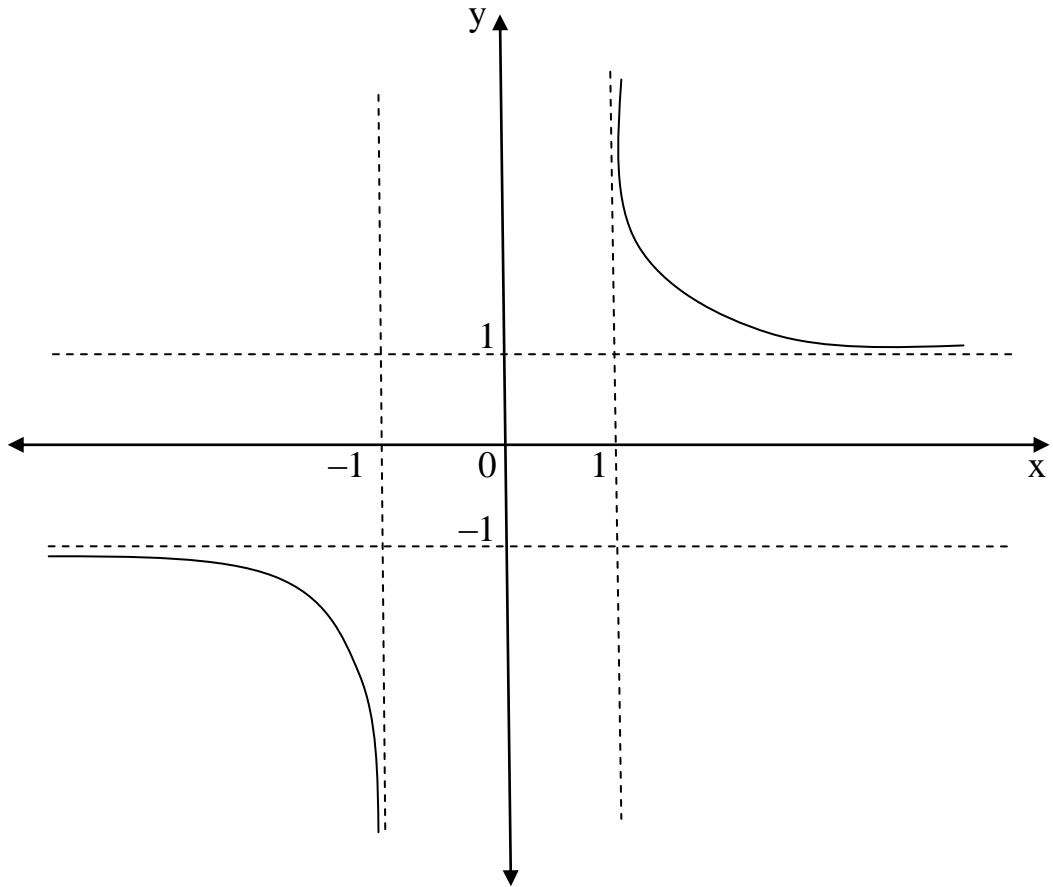
▪ Extrema

Since $f'(x) = \frac{\sqrt{x^2-1} \cdot 1 - x \cdot \frac{2x}{2\sqrt{x^2-1}}}{x^2-1} = \frac{1}{(x^2-1)\sqrt{x^2-1}} \neq 0$. No Extrema.

No Inflection points.

▪ It is odd function because $f(-x) = f(x)$.

See the figure.



(c)(ii) $g(x) = \frac{x}{1+x^2}$. Domain f is \mathbb{R} .

▪ Asymptotic lines

No vertical asymptotes.

Since $\lim_{x \rightarrow \pm\infty} \frac{x}{1+x^2} = 0$

Then, the line $y = 0$ is horizontal asymptote.

No inclined asymptotes.

▪ Points of intersection

With x – axis: solve $\frac{x}{1+x^2} = 0$, we get $x = 0$.

With y – axis: $f(0) = \frac{0}{1+0} = 0$, then $y = 0$.

▪ Extrema

Since $f'(x) = \frac{(1+x^2) \cdot 1 - x \cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = 0$. Then $1 - x^2 = 0$

We get the point $(1, 1/2)$ which is maximum and $(-1, -1/2)$ which is minimum.

▪ Inflection points

Since $f''(x) = \frac{(1+x^2)^2 \cdot -2x - (1-x^2) \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} = \frac{2x^3 - 6x}{(1+x^2)^3} = 0$

Then $2x^3 - 6x = 0$

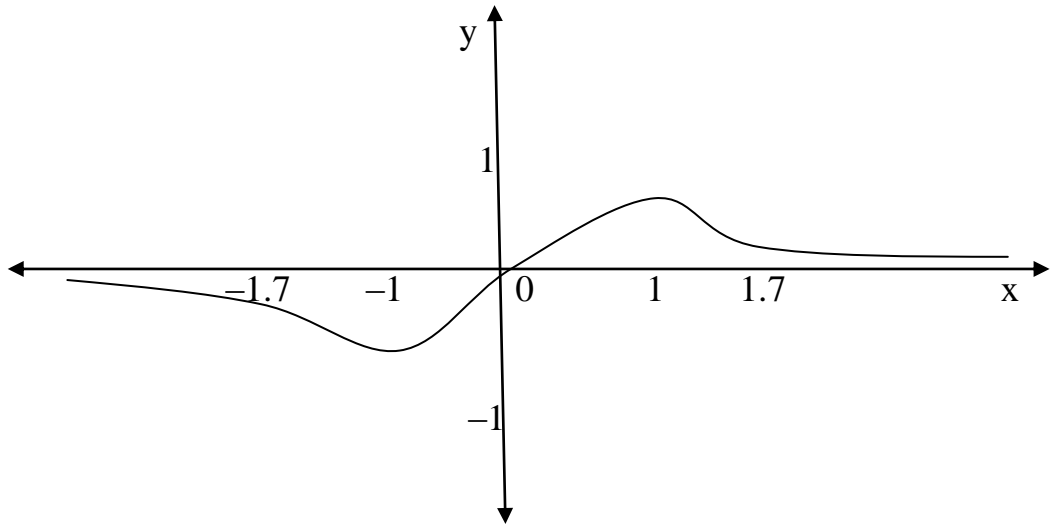
Then $x = 0, \sqrt{3} = 1.7, -\sqrt{3} = -1.7$

We get the inflection points:

$(0,0), \left(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{8}}\right) = (1.7, 0.6), \left(-\sqrt{3}, \frac{-\sqrt{3}}{\sqrt{8}}\right) = (-1.7, -0.6)$

▪ It is odd function because $f(-x) = f(x)$.

See the figure.



-----12-Marks

$$(d) \int (x^3 + 3^x) dx = \frac{x^4}{4} + \frac{3^x}{\ln 3} + c ,$$

$$\int (2^x - 3^x)^2 dx = \int (4^x + 9^x - 2 \cdot 6^x) dx = \frac{4^x}{\ln 4} + \frac{9^x}{\ln 9} - 2 \frac{6^x}{\ln 6} + c$$

$$\begin{aligned} \int \cos^5 2x dx &= \int \cos 2x [\cos^2 2x]^2 dx = \int \cos 2x [1 - \sin^2 2x]^2 dx \\ &= \int \cos 2x [1 - 2\sin^2 2x + \sin^4 2x] dx \\ &= \int (\cos 2x - 2 \cos 2x \sin^2 2x + \cos 2x \sin^4 2x) dx \\ &= \frac{1}{2} \sin 2x - \frac{1}{3} \sin^3 2x + \frac{1}{10} \sin^5 2x + c \end{aligned}$$

$$\int_0^{\pi} \sin^{10} x dx = 2 \int_0^{\pi/2} \sin^{10} x dx = 2 \frac{9.7.5.3.1}{10.8.6.4.2} \cdot \frac{\pi}{2}$$

-----10-Marks

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Answer of Question 3

(a) Find the sum of the series $\sum_{r=1}^n \frac{1}{(2r-1)(2r+1)}$

Solution:

$$u_r = \frac{1}{(2r-1)(2r+1)} = \frac{1}{2} \left[\frac{1}{(2r-1)} + \frac{1}{(2r+1)} \right] = f(r) - f(r+1)$$

$$S_n = f(1) - f(n+1) = \frac{1}{2} \left[\frac{1}{(2-1)} + \frac{1}{(2n+1)} \right] = \frac{n}{2n+1}$$

$$\boxed{\sum_{r=1}^n \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}}$$

-----10-Marks

(b) Expand the fraction $\frac{(3x+5)}{x^3-x^2-x+1}$ into ascending power series of x .

Solution:

$$\begin{aligned} x^3 - x^2 - x + 1 &= (x^3 - x^2) - (x - 1) = x^2(x - 1) - (x - 1) \\ &= (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1) = (x - 1)^2(x + 1) \end{aligned}$$

$$\text{Then } \frac{(3x+5)}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \quad (1)$$

Then we have

$$(3x+5) = A(x-1)^2 + B(x-1)(x+1) + C(x+1) \quad (2)$$

For $x = 1 \Rightarrow 2C = 8 \Rightarrow \boxed{C = 4}$

For $x = -1 \Rightarrow 4A = 2 \Rightarrow \boxed{A = 1/2}$

And for $x = 0 \Rightarrow A - B + C = 5 \Rightarrow \boxed{B = -1/2}$

$$\therefore \frac{(3x + 5)}{x^3 - x^2 - x + 1} = \frac{1/2}{x + 1} + \frac{-1/2}{x - 1} + \frac{4}{(x - 1)^2}$$

$$\begin{aligned} \therefore \frac{(3x + 5)}{x^3 - x^2 - x + 1} &= \frac{1/2}{x + 1} + \frac{-1/2}{x - 1} + \frac{4}{(x - 1)^2} \\ &= \frac{1}{2}(1 + x)^{-1} - \frac{1}{2}(1 - x)^{-1} + 4(1 - x)^{-2} \end{aligned}$$

$$\frac{(3x + 5)}{x^3 - x^2 - x + 1} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n x^n - \frac{1}{2} \sum_{n=0}^{\infty} x^n + 4 \sum_{n=0}^{\infty} (n + 1)x^n$$

-----10-Marks

(c) Find the real and imaginary part to $\sin z$ and e^{iz} for any complex number z

Solution

$$\begin{aligned} \sin z &= \sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy) \\ &= \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

The real part is **$\sin x \cosh y$**

The imaginary part is **$\cos x \sinh y$**

$$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y} e^{ix} = e^{-y} (\cos x + i \sin x)$$

The real part is $e^{-y} \cos x$ and the imaginary part is $e^{-y} \sin x$

-----5-Marks

Answer of Question 4

(a) Gauss Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 2 & 1 & 0 & 8 \\ 1 & 2 & 2 & 9 \end{array} \right] \approx \left[\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 0 & -1 & -4 & -6 \\ 0 & 1 & 0 & 2 \end{array} \right] \approx \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & -1 & -4 & -6 \\ 0 & 0 & -4 & -4 \end{array} \right] \approx \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & -1 & -4 & -6 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right] \text{ then } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

-----10-Marks

(b) Find the eigen values and eigen vectors for the matrix

$$A = \begin{bmatrix} -2 & -2 & -4 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

Solution:

The characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} -2 - \lambda & -2 & -4 \\ 2 & 3 - \lambda & 2 \\ 3 & 2 & 5 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

then the characteristic roots are $\lambda = 1, 2, 3$

to determine the eigen vectors we use the equation

$$(A - \lambda I)X_1 = \mathbf{0} \text{ where } X_1 = (a_1, a_2, a_3)$$

at $\lambda = 1$ we have

$$(A - \lambda I)X'_1 = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{0}$$

this system corresponds to

$$a_1 + 2a_3 = 0, a_2 - a_3 = 0 \quad \text{let } a_3 = t \neq 0$$

Then $X_1 = (-2t, t, t)$ if $t = 1$ then $X_1 = (-2, 1, 1)$

at $\lambda = 2$ let $X_2 = (b_1, b_2, b_3)$ we and

$$(A - \lambda I)X'_2 = \begin{bmatrix} -4 & -2 & -4 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{0}$$

Then $b_1 + b_3 = 0, b_2 = 0$ and $X_2 = (-1, 0, 1)$

When $\lambda = 3$ let $X_3 = (c_1, c_2, c_3)$ and

$$(A - \lambda I)X'_3 = \begin{bmatrix} -5 & -2 & -4 \\ 2 & 0 & 2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$$

then $c_1 + c_3 = 0, 2c_2 - c_3 = 0$ $X_3 = (-2, 1, 2)$

-----10-Marks

(c) Solve $12x^3 - 8x^2 + kx + 18 = 0$ and find the value of k given that sum of two roots equal zero.

Solution

Let the roots are a, b and c

The sum of two roots equal zero we put $a + b = 0$

The relation between the roots and coefficients

$$a + b + c = \frac{8}{12} = \frac{2}{3} \text{ and since } a + b = 0 \text{ then } c = \frac{2}{3}$$

$$abc = \frac{-18}{12} = \frac{-3}{2} \text{ and since } c = \frac{2}{3} \text{ then } ab = \frac{-9}{4}$$

$$a \text{ and } c \text{ are roots for } x^2 - \frac{9}{4} = 0 \text{ then } a = \frac{3}{2} \text{ and } c = \frac{-3}{2}$$

$$\text{and } k = 12(ab + ac + bc)$$

-----5-Marks

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